

NON-GENERICITY OF INFINITESIMAL VARIATIONS OF HODGE STRUCTURES ARISING IN SOME GEOMETRIC CONTEXTS

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ABSTRACT. We prove that the infinitesimal variations of Hodge structure arising in a number of geometric situations are non-generic. In particular, we consider the case of generic hypersurfaces in complete smooth projective toric varieties, generic hypersurfaces in weighted projective spaces and generic complete intersections in projective space and show that, for sufficiently high degrees, the corresponding infinitesimal variations are non-generic.

1. INTRODUCTION

A variation of Hodge structure can be described, using the language of exterior differential systems, as an integral manifold of Griffiths' differential system over the period domain. An important problem in Hodge theory is the study of the geometric locus, that is, the locus of those variations of Hodge structure that arise from the cohomology of a family of polarized projective varieties.

An infinitesimal version of this problem consists of describing the infinitesimal variations of Hodge structure —the integral elements of Griffiths' system— that arise from geometric variations. In [1] the first author has shown that the infinitesimal variations arising from deformations of hypersurfaces of sufficiently high degree in projective space are non-generic in the space of infinitesimal variations. The purpose of the present paper is to show that this property holds in a variety of geometric situations, namely, generic hypersurfaces of complete smooth toric varieties (Theorem 3.12), generic hypersurfaces in weighted projective space (Theorem 4.5) and generic complete intersections in projective space (Theorem 5.6). In all cases, requirements of sufficiently high degree apply.

Considering all these results we begin to see a general principle that infinitesimal variations of geometric origin (eventually satisfying some condition analogous to high degree) are non-generic.

The main tools used in this paper are the appropriate residue theories for simplicial toric varieties and for complete intersections in projective space, as well as infinitesimal Torelli theorems, dualities and Macaulay's theorem.

The plan for the paper is as follows: in Section 2 we review some results from [1] and describe an approach to proving the non-genericity of families of infinitesimal variations. In Section 3 we study the infinitesimal variations associated to generic hypersurfaces in complete simplicial projective toric varieties and reduce the proof of non-genericity results to a numerical condition strongly related to infinitesimal Torelli theorems (Theorem 3.9). One such Torelli theorem, due to M. Green, allows us to conclude the non-genericity in the case of smooth ambient spaces. In Section 4 we specialize the toric analysis to the case where the ambient space is a weighted

projective space where the numerical condition now follows from a result of L. Tu. Finally, in Section 5, the general approach is specialized to the case of complete intersections.

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2. THE PROJECTION OF THE INTEGRAL ELEMENTS OF THE GRIFFITHS' SYSTEM TO A GRASSMANNIAN, THE SYMMETRIZERS CORRESPONDENCE

Let us fix some notation (see [11] and [13] for more details). Recall that a (real) Hodge structure of weight k on the real vector space H with Hodge numbers $h^{k,0}, \dots, h^{0,k}$, consists of a grading $H \otimes \mathbb{C} = H^{k,0} \oplus \dots \oplus H^{0,k}$ such that $\dim H^{j,k-j} = h^{j,k-j}$ and $\overline{H^{j,k-j}} = H^{k-j,j}$. We will usually denote H and its complexification by H . We define $H^q := \text{hom}(H^{k-q,q}, H^{k-q-1,q+1})$, and consider $\oplus_{0 \leq q \leq k-1} H^q$ as a subset of $\text{hom}(H, H)$. We also define the following maps for $0 \leq q \leq k-1$:

$$p_q : \oplus_{0 \leq r \leq k-1} H^r \longrightarrow H^q$$

$$\alpha \longmapsto \alpha|_{H^{k-q,q}}$$

Remark 2.1. p_q is the natural projection of $\oplus_{0 \leq a \leq k-1} H^a$ onto H^q .

The periods space (i.e. the set of all *polarized* Hodge structures with fixed Hodge numbers and polarization Q , see [11]) is the homogeneous variety $\mathcal{D} \simeq G/P$ (with $G = SO(H, Q)$ and P a parabolic subgroup). We note $\mathfrak{g}, \mathfrak{p}$ the Lie algebras of G and P . Then \mathfrak{g} is given by:

$$\mathfrak{g} = \{X \in \text{End}(H) \mid Q(Xv, w) + Q(v, Xw) = 0, \forall v, w \in H\} \quad (2.1)$$

Fixing a reference structure $H_0 := \{H_0^{k-q,q}\} \in \mathcal{D}$, we can also consider the following subspaces of $\text{End}(H)$:

$$\text{End}(H)^{p,-p} = \{X \in \text{End}(H) \mid \forall r+s=k, X(H_0^{r,s}) \subset H_0^{r+p,s-p}\} \quad (2.2)$$

and then define $\mathfrak{g}^{p,-p} := \mathfrak{g} \cap \text{End}(H)^{p,-p}$. We also note

$$\mathfrak{g}^0 := \mathfrak{g}^{0,0}, \quad \mathfrak{g}^- := \bigoplus_{p < 0} \mathfrak{g}^{p,-p}, \quad \mathfrak{g}^+ := \bigoplus_{p > 0} \mathfrak{g}^{p,-p}$$

So that we have $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+$. Moreover we have that $\mathfrak{p} = \mathfrak{g}^0 \oplus \mathfrak{g}^+$ and, as $\mathcal{D} \simeq G/P$, we conclude that

$$T_{H_0} \mathcal{D} \simeq \mathfrak{g}^-.$$

Let $V := \mathfrak{g}^{-1,1} \subset \oplus_{0 \leq a \leq k-1} H^a$, and let V_d denote the set of infinitesimal variations of Hodge structure of dimension d , then $V_d \subset G(d, V)$. In fact, V_d is the algebraic subvariety of $G(d, V)$ of all abelian d -subalgebras of V . Moreover we have the following:

Lemma 2.2. *For all $i \in \{0, \dots, k\}$, $p_i : G(d, V) \rightarrow G(d, H^i)$ is a rational map.*

We now recall the definition of Symmetrizer due to R. Donagi:

Definition 2.3. Let $\psi : E \times F \rightarrow G$ be a bilinear map. We define

$$\text{Symm } \psi := \{q \in \text{hom}(E, F) \mid \forall \alpha, \alpha' \in E, \psi(\alpha, q(\alpha')) = \psi(\alpha', q(\alpha))\}$$

The link between the infinitesimal variations of Hodge structure and the symmetrizers is given by the following proposition ([1], Proposition 3.2):

Proposition 2.4. *For any $E^0 \in G(d, H^0)$, we define the following bilinear map:*

$$\begin{aligned} \phi_{E^0} : E^0 \times H^1 &\longrightarrow \text{hom}(H^{k,0}, H^{k-2,2}) \\ (\alpha, \beta) &\longmapsto \beta \circ \alpha \end{aligned}$$

then we have for $k \geq 3$

$$p_1(p_0^{-1}(E^0) \cap V_d) \simeq \text{Symm } \phi_{E^0}$$

We need one more proposition to explain the principle of “non-genericity” for infinitesimal variations of Hodge structure.

Proposition 2.5. *Let G^0, G^1, G^2 be three \mathbb{C} -vector spaces with $\dim G^0 > 1$. For all $E^0 \subset \text{hom}(G^0, G^1)$, we consider the following bilinear map :*

$$\begin{aligned} \phi_{E^0} : E^0 \times \text{hom}(G^1, G^2) &\longrightarrow \text{hom}(G^0, G^2) \\ (\alpha, \beta) &\longmapsto \beta \circ \alpha \end{aligned}$$

Let d be a positive integer, $d \leq \dim \text{hom}(G^0, G^1)$; if we have the following inequality

$$d \geq 3 \left(\left\lfloor \frac{\dim G^1 - 1}{\dim G^0} \right\rfloor + 1 \right) \quad (2.3)$$

then any generic $E^0 \in G(d, \text{hom}(G^0, G^1))$ satisfies

$$\text{Symm}(\phi_{E^0}) = \{0\}$$

Proof. ([1], Proposition 3.3) The space of symmetrizers $\text{Symm}(\phi_{E^0})$ is defined by a set of linear equations depending on $E^0 \in G(d, \text{hom}(G^0, G^1))$; then the condition $\text{Symm}(\phi_{E^0}) = \{0\}$ is open in $G(d, \text{hom}(G^0, G^1))$ and the proof amounts to build a \mathbb{C} -vector space in $\text{hom}(G^0, G^1)$ for which the rank of the linear equations defining the symmetrizers is maximal (the dimensions are such that maximal rank for those equations implies triviality of their zero set). \square

In our context we will have $G^i := H^{k-i,i}$ ($0 \leq i \leq 2$), and $E^0 := p_0(E)$ where E will be an infinitesimal variation of Hodge structure. In this setting the inequality (2.3) now reads:

$$\dim p_0(E) \geq 3 \left(\left\lfloor \frac{h^{k-1,1} - 1}{h^{k,0}} \right\rfloor + 1 \right) \quad (2.4)$$

Now we can state the main theorem that we will use in our geometric applications.

Theorem 2.6. *Let E be an infinitesimal variation of Hodge structure such that $\dim p_0(E) = \dim E$, $p_1(E) \neq \{0\}$ and E satisfies inequality (2.4) then E is non-generic, that is, E must lie in a proper subvariety of $V_{\dim E}$.*

Proof. Define $E^0 := p_0(E)$ and $d := \dim E$. Using Proposition 2.4 we have that $p_1(E) \in p_1(p_0^{-1}(E^0) \cap V_d) \simeq \text{Symm } \phi_{E^0}$, and as we made the assumption that $p_1(E) \neq \{0\}$, this implies that $\text{Symm } \phi_{E^0} \neq \{0\}$. But as E satisfies (2.4), the generic element $X \in G(d, H^0)$ satisfies $\text{Symm } \phi_X = \{0\}$, then $p_0(E)$ must lie in a proper subvariety S of $G(\dim p_0(E), H^0)$; therefore E lies in $p_0^{-1}(S) \cap V_{\dim E}$ which is a proper subvariety of V_d (because p_0 is regular and surjective). \square

The following are applications of Theorem 2.6; they are the main results of this paper.

Theorem 2.7. *Let P be a complete smooth projective toric variety of dimension $n \geq 4$ and $\beta \in A_{n-1}(P)$ the degree of an ample Cartier divisor. Let E be the infinitesimal variation of Hodge structure associated to a generic $f \in S_{t\beta}$, where S is the homogeneous coordinate ring of P . Then, there exists $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$, E is non-generic.*

Theorem 2.8. *Let $P = \mathbb{P}^n(q_0, \dots, q_n)$ be the weighted projective space with $n \geq 4$, $q_0 = \gcd(q_1, \dots, q_n) = 1$. Then, if $\text{lcm}(q_1, \dots, q_n) \mid d$ and $\text{lcm}(q_1, \dots, q_n) \mid \sum_{j=0}^n q_j$, there is a degree d_0 such that for all $d \geq d_0$ the infinitesimal variation of Hodge structure associated to a generic hypersurface of degree d is non-generic.*

Theorem 2.9. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection defined by the intersection of c hypersurfaces of degrees d_1, \dots, d_c . If $\dim X \geq 3$ there exists $K \in \mathbb{N}$ such that if the degree of the canonical bundle $d(X) = \sum_{a=1}^c d_a - (n+1) \geq K$, then the infinitesimal variation of Hodge structure associated to the deformation of X , E , is non-generic in $V_{\dim E}$.*

Remark 2.10. These theorems state asymptotic results. Nevertheless, for concrete examples the conditions on the degrees of the hypersurfaces that ensure non-genericity can be stated and tested explicitly. For the last Theorem, effective bounds are also included.

3. HYPERSURFACES IN TORIC VARIETIES

In this section we describe the infinitesimal variation of Hodge structure associated to the family of hypersurfaces of fixed degree in a complete simplicial projective toric variety P . Using this description the non-genericity of the infinitesimal variation for hypersurfaces of high degree can be established modulo some algebraic conditions. Finally, we show that the algebraic conditions hold, for instance, when the hypersurfaces and P are smooth and, also, when P is a weighted projective space.

Let P be the n -dimensional complete simplicial toric variety defined by the fan $\Sigma \subset N$, where N is a lattice of rank n in \mathbb{R}^n . P can be described as a geometric quotient as follows [5]. Let $S := \mathbb{C}[z_1, \dots, z_r]$ be the polynomial ring with variables corresponding to the integer generators n_j of the 1-dimensional cones of Σ . Let $Z := \bigcap_{\sigma \subset \Sigma} \{z \in \mathbb{C}^r : \hat{z}_\sigma = 0\}$, where for each cone $\sigma \subset \Sigma$, $\hat{z}_\sigma := \prod_{n_j \notin \sigma} z_j$. Then P is the geometric quotient of $U := \mathbb{C}^r \setminus Z$ by the algebraic group $G := \text{hom}(A_{n-1}(P), \mathbb{C}^*)$, where $A_{n-1}(P)$ is the Chow group of P .

If M is the lattice dual to N , a grading on S can be induced from the exact sequence $0 \rightarrow M \rightarrow \mathbb{Z}^r \rightarrow A_{n-1}(P) \rightarrow 0$, where the second arrow is $m \mapsto (\langle m, n_1 \rangle, \dots, \langle m, n_r \rangle)$ and the third is $(a_1, \dots, a_r) \mapsto a_1[D_1] + \dots + a_r[D_r]$, where D_j is the divisor associated to the generator n_j . With this notation, $\deg(z_1^{a_1} \dots z_r^{a_r}) := a_1[D_1] + \dots + a_r[D_r] \in A_{n-1}(P)$. This grading coincides with the one induced by the action of G on \mathbb{C}^r . The graded piece of S of degree $\beta \in A_{n-1}(P)$ is denoted by S_β .

Given $f \in S_\beta$, let $V(f) := \{z \in \mathbb{C}^r : f(z) = 0\}$; $V(f) \cap U$ is G stable and hence descends to a hypersurface $X_f \subset P$. Since Σ is simplicial, P is a V -manifold. Moreover, by Proposition 4.15 in [4], if β is Cartier and ample, for generic $f \in S_\beta$, X_f is a V -submanifold of P . Let $B_\beta \subset S_\beta$ be the Zariski open of those f for which X_f is a V -submanifold of P and let $\pi : \mathcal{X} \rightarrow B_\beta$ be the family of hypersurfaces $\mathcal{X}_f = X_f \subset P$. If we assume that P is a projective variety, classical results of P.

Deligne show that $R^{n-1}\pi_*\mathbb{Q}$ defines a graded-polarized variation of mixed Hodge structure. On the other hand, by [17, §1], it is known that since X_f is a V -manifold, $H^{n-1}(X_f, \mathbb{Q})$ carries a pure Hodge structure of weight $n-1$. Furthermore, if $\omega := c_1(\mathcal{O}_P(1))$ and L_ω is the operator induced by left multiplication by ω ,

$$H^{n-1}(X_f)_p := \ker(L_\omega : H^{n-1}(X_f, \mathbb{Q}) \rightarrow H^{n+1}(X_f, \mathbb{Q})),$$

the *primitive cohomology*, is a sub Hodge structure that is polarized. In what follows we will consider a sub Hodge structure of $H^{n-1}(X_f)_p$, namely, the *vanishing cohomology* (also known as *variable cohomology*) defined as

$$H^{n-1}(X_f)_v := \ker(j_* : H^{n-1}(X_f, \mathbb{Q}) \rightarrow H^{n+1}(X_f, \mathbb{Q})),$$

where j_* is the Gysin morphism associated to the inclusion $j : X_f \rightarrow P$. Notice that in [4] this vanishing cohomology is called primitive cohomology. Since $L_\omega = j^* \circ j_*$ it is clear that $H^{n-1}(X_f)_v$ is a sub Hodge structure of $H^{n-1}(X_f)_p$, that is polarized, hence, $H^{n-1}(X_f)_v$ is a polarized sub Hodge structure of $H^{n-1}(X_f)_p$ of weight $n-1$. To sum up, we have:

Proposition 3.1. *For $\beta \in A_{n-1}(P)$ Cartier and ample, the vanishing cohomology of the family of hypersurfaces $\pi : \mathcal{X} \rightarrow B_\beta$ defines the polarized variation of Hodge structure of weight $n-1$, whose fibers are $H^{n-1}(X_f, \mathbb{Q})_v$.*

If \mathcal{D} is the classifying space for the Hodge structures described in Proposition 3.1 and Γ is the monodromy group, the previous result defines a *period mapping* $\Phi : B_\beta \rightarrow \mathcal{D}/\Gamma$. In fact, upon restriction to a Zariski open $B'_\beta \subset B_\beta$, Φ factors through the *generic coarse moduli space for hypersurfaces of P with degree β* , $M_\beta := B'_\beta / \widetilde{\text{Aut}_\beta}$, as defined in [4, §13]:

$$\begin{array}{ccc} B'_\beta & \xhookrightarrow{\quad} & B_\beta \xrightarrow{\quad \Phi \quad} \mathcal{D}/\Gamma \\ \kappa \downarrow & \nearrow \Psi & \\ M_\beta & & \end{array} \quad (3.1)$$

As noted in [4], for B'_β sufficiently small, κ is a smooth map. Taking differentials at the point $f \in B'_\beta$ we have $d\Phi_f = d\Psi_{[f]} d\kappa_f$, with $d\kappa_f$ onto. Notice that the horizontality of Φ implies that of Ψ . In what follows we will consider the infinitesimal variation of Hodge structure E_f associated to Ψ , $d\Psi_{[f]} : T_{[f]}M_\beta \rightarrow T_{\Psi([f])}\mathcal{D}$. In fact, we will be only interested in the image of $d\Phi_f$, that we will continue to call E_f . Since $d\kappa_f$ is onto, E_f is also the image of $d\Psi_{[f]}$.

The following result shows some relations between the cohomology of a hypersurface X_f in a toric variety and graded parts of the Jacobian ring of f .

Theorem 3.2. *Let P be a complete simplicial projective toric variety of dimension $n \geq 4$, and $X_f \subset P$ a quasi-smooth ample hypersurface defined by $f \in S_\beta$ that defines a Cartier divisor. Let $R(f) := S/\langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_r} \rangle$ be the Jacobian ring of f , with the grading inherited from S . Then,*

- (1) $H^0(P, \mathcal{O}_P(X_f)) \simeq S_\beta$. The following diagram is commutative up to a multiplicative constant.

$$\begin{array}{ccc} H^0(P, \mathcal{O}_P(X_f)) \otimes H^0(P, \Omega_P^n(pX_f)) & \longrightarrow & H^0(P, \Omega_P^n((p+1)X_f)) \\ \downarrow & & \downarrow \\ T_f B_\beta \otimes H^{n-p, p-1}(X_f)_v & \longrightarrow & H^{n-p-1, p}(X)_v \end{array} \quad (3.2)$$

where the top arrow is induced by the product of sections and the bottom by the infinitesimal variation of Hodge structure, and the vertical arrows are the surjective maps

$$\bar{\alpha}_p : H^0(P, \Omega_P^n(pX_f)) \rightarrow H^{n-p, p-1}(X_f)_v$$

induced by the residue.

- (2) The diagram

$$\begin{array}{ccc} T_{[f]} M_\beta & \xrightarrow{d\Psi_{[f]}} & \text{hom}(H^{n-1, 0}(X_f)_v, H^{n-2, 1}(X_f)_v) \\ \simeq \downarrow & & \downarrow \simeq \\ R_\beta & \xrightarrow{\times} & \text{hom}(R(f)_{\beta-\beta_0}, R(f)_{2\beta-\beta_0}) \end{array} \quad (3.3)$$

is commutative up to a multiplicative constant. Here $\beta_0 := \deg(z_1 \dots z_r) \in A_{n-1}(P)$.

Proof. The first assertion of (1) is Lemma 4.11 in [4]. Theorem 6.13 from [20] (modified for toric varieties using the residue theory developed in [4]) gives that the diagram

$$\begin{array}{ccc} H^0(P, \Omega_P^n(pX_f)) & \longrightarrow & \text{hom}(H^0(P, \mathcal{O}_P(X_f)), H^0(P, \Omega_P^n((p+1)X_f))) \\ \downarrow & & \downarrow \\ H^{n-p, p-1}(X_f)_v & \longrightarrow & \text{hom}(T_f B_\beta, H^{n-p-1, p}(X_f)_v) \end{array} \quad (3.4)$$

is commutative up to a multiplicative constant. The vertical arrows are induced by the surjective residue maps $\bar{\alpha}_p$ and the isomorphism $H^0(P, \mathcal{O}_P(X_f)) \simeq S_\beta \simeq T_f B_\beta$. From (3.4) the rest of part (1) follows.

To prove part (2), we start by showing that the vertical arrows are isomorphisms. Proposition 13.7 in [4] does it for the left arrow, while Theorem 10.13 in [4] implies the result for the right one.

Next we establish the commutativity of diagram (3.3). Starting from (3.4) and using Lemma 4.11 and Theorem 9.7 of [4], we have that

$$\begin{array}{ccc} S_{p\beta-\beta_0} & \longrightarrow & \text{hom}(S_\beta, S_{(p+1)\beta-\beta_0}) \\ \simeq \downarrow & & \downarrow \simeq \\ H^0(P, \Omega_P^n(pX_f)) & \longrightarrow & \text{hom}(H^0(P, \mathcal{O}_P(X_f)), H^0(P, \Omega_P^n((p+1)X_f))) \\ \downarrow & & \downarrow \\ H^{n-p, p-1}(X_f)_v & \longrightarrow & \text{hom}(T_f B_\beta, H^{n-p-1, p}(X_f)_v) \end{array}$$

commutes up to a multiplicative constant. From here it is easy to see that the same holds for

$$\begin{array}{ccc} R_{p\beta-\beta_0} & \longrightarrow & \text{hom}(S_\beta, R_{(p+1)\beta-\beta_0}) \\ \simeq \downarrow & & \downarrow \simeq \\ H^{n-p,p-1}(X_f)_v & \longrightarrow & \text{hom}(T_f B_\beta, H^{n-p-1,p}(X_f)_v) \end{array}$$

and, eventually, for

$$\begin{array}{ccc} S_\beta & \longrightarrow & \text{hom}(R_{p\beta-\beta_0}, R_{(p+1)\beta-\beta_0}) \\ \simeq \downarrow & & \downarrow \simeq \\ T_f B_\beta & \longrightarrow & \text{hom}(H^{n-p,p-1}(X_f)_v, H^{n-p-1,p}(X_f)_v) \end{array}$$

and, then, (3.3) follows by noticing that the horizontal arrows factor through R_β and $T_{[f]}M$ respectively. \square

V. Batyrev gives a similar description of the differential of the period mapping in his work on affine hypersurfaces (see Proposition 11.8 in [3]).

To state the following result we will write $O(t^k)$ to denote a function—a polynomial in this case—with $\frac{O(t^k)}{t^k}$ bounded as $t \rightarrow \infty$. Also, if $M \simeq \mathbb{Z}^n \subset \mathbb{R}^n$ is a lattice, $\text{vol}(\pi)$ is the volume of the lattice polytope $\pi \subset \mathbb{R}^n$, normalized so that the unit n -cube of the lattice M is 1.

Lemma 3.3. *Let D be an ample Cartier divisor in the n -dimensional complete simplicial toric variety P and $\Delta \subset \mathbb{R}^n$ its associated polytope. If X is a generic ample hypersurface of degree $[tD] \in A_{n-1}(P)$ for $t \in \mathbb{N}$, then $\text{vol}(\Delta) > 0$ and*

- (1) $h_t^{n-1,0} := \dim H^{n-1,0}(X) = \text{vol}(\Delta)t^n + O(t^{n-1})$,
- (2) $h_t^{n-2,1} := \dim H^{n-2,1}(X) = (2^n - (n+1))\text{vol}(\Delta)t^n + O(t^{n-1})$ and
- (3) $\mu_t := \dim M_{[tD]} \geq \text{vol}(\Delta)t^n + O(t^{n-1})$.

Proof. Let $\Delta_t := t\Delta$ be the polytope in $M \otimes \mathbb{R} \simeq \mathbb{R}^n$ defined by tD . By (5.5) in [6] we have

$$\begin{aligned} h_t^{n-1,0} &:= \dim H^{n-1,0}(X) = l^*(\Delta_t) \\ h_t^{n-2,1} &:= \dim H^{n-2,1}(X) = l^*(2\Delta_t) - (n+1)l^*(\Delta_t) - \sum_{\Gamma \in \mathcal{F}_{n-1}(\Delta_t)} l^*(\Gamma), \end{aligned} \quad (3.5)$$

where $\mathcal{F}_{n-1}(\Delta_t)$ is the set of $(n-1)$ -dimensional faces of Δ_t and $l^*(\pi)$ denotes the number of lattice points that lie in the relative interior of the polytope π .

The number of lattice points in a lattice polytope $\pi_t := t\pi \subset \mathbb{R}^n$ is $E_\pi(t) := l(\pi_t)$, where $l(\pi_t)$ is the number of lattice points in π_t and E_π is the Ehrhart polynomial of π (see [8]). E_π is a polynomial of degree at most n where the coefficient of t^n is $\text{vol}(\pi)$. Furthermore, the reciprocity law says that $E_\pi(-t) = (-1)^n l^*(\pi_t)$.

Since D is an ample Cartier divisor, Δ is an n -dimensional lattice polytope [15, §2.2] and we have $l^*(\Delta_t) = (-1)^n E_\Delta(-t) = \text{vol}(\Delta)t^n + O(t^{n-1})$ with $\text{vol}(\Delta) > 0$. Hence

$$h_t^{n-1,0} = \text{vol}(\Delta)t^n + O(t^{n-1}).$$

Similarly,

$$\begin{aligned}
h_t^{n-2,1} &= l^*(2\Delta_t) - (n+1)l^*(\Delta_t) - \sum_{\Gamma \in \mathcal{F}_{n-1}(\Delta_t)} l^*(\Gamma) \\
&= (-1)^n E_{2\Delta}(-t) - (n+1)(-1)^n E_{\Delta}(-t) - \sum_{\Gamma \in \mathcal{F}_{n-1}(\Delta_t)} (-1)^{n-1} E_{\Gamma}(-t) \\
&= \text{vol}(2\Delta)t^n - (n+1)\text{vol}(\Delta)t^n - \sum_{\Gamma \in \mathcal{F}_{n-1}(\Delta_t)} \text{vol}(\Gamma)t^{n-1} + O(t^{n-1}) \\
&= (2^n - (n+1))\text{vol}(\Delta)t^n + O(t^{n-1}).
\end{aligned}$$

Now we turn to μ_t . By Theorem 3.2 we have $\mu_t = \dim R(f)_{[tD]}$ for any generic polynomial $f \in S_{[tD]}$. Since $\dim R(f)_{[tD]} = \dim S_{[tD]} - \dim J(f)_{[tD]}$, we study each term separately. Writing $D = \sum_{j=1}^r b_j D_j$ where the D_j are the torus-invariant divisors associated to the one dimensional cones in the fan of P , we have

$$\begin{aligned}
\dim S_{[tD]} &= \#\{(a_1, \dots, a_r) \in \mathbb{Z}_{\geq 0}^r : \deg(z_1^{a_1} \cdots z_r^{a_r}) = [tD]\} \\
&= \#\{m \in M : \langle m, n_j \rangle + tb_j \geq 0 \text{ for } j = 1, \dots, r\} \\
&= \#(\Delta_t \cap M) = l(\Delta_t),
\end{aligned}$$

Also, since $J(f)_{[tD]} = \{\sum_{j=1}^r g_j(z) \frac{\partial f}{\partial z_j} : g_j \in S_{[D_j]}, \forall j\}$, we have $\dim J(f)_{[tD]} \leq K$, for $K := \sum_{j=1}^r \dim S_{[D_j]}$. Notice that K is independent of t (in fact, it is determined by the 1-dimensional cones of the fan of P). We conclude then that

$$\begin{aligned}
\mu_t &= \dim R(f)_{[tD]} = \dim S_{[tD]} - \dim J(f)_{[tD]} \geq l(\Delta_t) - K \\
&= \text{vol}(\Delta)t^n + O(t^{n-1}).
\end{aligned}$$

□

Proposition 3.4. *Let $\beta \in A_{n-1}(P)$ be the class of an ample Cartier divisor in the complete simplicial toric variety P of dimension $n \geq 4$. Then, there exists $t_0 \in \mathbb{N}$ such that the cohomology of the generic ample hypersurface X of degree $t\beta$ for $t \geq t_0$ satisfies*

$$\mu_t \geq 3 \left(\left\lceil \frac{h_v^{n-2,1}(X) - 1}{h_v^{n-1,0}(X)} \right\rceil + 1 \right), \quad (3.6)$$

where $\mu_t := \dim M_{t\beta}$.

Proof. Let D be an ample Cartier divisor with $[D] = \beta$. Then, by Lemma 3.3 we have

$$\begin{aligned}
\left\lceil \frac{h_t^{n-2,1} - 1}{h_t^{n-1,0}} \right\rceil &\leq \frac{h_t^{n-2,1} - 1}{h_t^{n-1,0}} \leq \frac{(2^n - (n+1))\text{vol}(\Delta)t^n + O(t^{n-1})}{\text{vol}(\Delta)t^n + O(t^{n-1})} \\
&= \frac{(2^n - (n+1))\text{vol}(\Delta) + \frac{O(t^{n-1})}{t^n}}{\text{vol}(\Delta) + \frac{O(t^{n-1})}{t^n}}
\end{aligned}$$

and, since the last expression converges to $2^n - (n+1)$ as $t \rightarrow \infty$, we see that the whole expression is bounded from above.

From

$$H^{n-1}(X_f, \mathbb{Q})_p = H^{n-1}(X_f, \mathbb{Q})_v \oplus j^* H^{n-1}(P, \mathbb{Q})$$

([20, Prop. 2.27]), Lefschetz's theorem and Bott's formula (see [12, Thm. 2.14]), it is easy to conclude that, for $n \geq 4$, $H^{n-1,0}(X) = H^{n-1,0}(X)_v$ and $H^{n-2,1}(X) =$

$H^{n-2,1}(X)_v$. Hence the Hodge numbers $h_t^{n-1,0}$ and $h_t^{n-2,1}$ compute the corresponding dimensions of the vanishing cohomology of X .

Thus, the right hand side of (3.6) is bounded from above. On the other hand, also from Lemma 3.3, we have $\mu_t \geq \text{vol}(\Delta)t^n + O(t^{n-1})$ with $\text{vol}(\Delta) > 0$, so that the left hand side of (3.6) grows like t^n , and the inequality holds for t sufficiently large. \square

Remark 3.5. In the proof of Proposition 3.4 a choice of divisor D is used. This choice determines how large t should be so that (3.6) holds. But any other divisor D' with the same degree would have an associated polytope Δ' that is a lattice translate of that of D ; in particular, this implies that they have the same number of lattice points and hence the left and right sides of (3.6) would be the same for D and D' . Hence, a value of t that makes (3.6) true for D , works for any other D' with degree β .

Next we study the p_1 projection of the infinitesimal variation of Hodge structure associated to sufficiently ample hypersurfaces in toric varieties. We first recall, in a slightly adapted form, Lemma 1.28 of [10]:

Lemma 3.6. *Let P be a projective variety and $\mathcal{E}_1, \mathcal{E}_2$ two coherent sheaves over P . For \mathcal{L} a sufficiently ample invertible sheaf, the multiplication map*

$$H^0(P, \mathcal{E}_1 \otimes \mathcal{L}^a) \otimes H^0(P, \mathcal{E}_2 \otimes \mathcal{L}^b) \rightarrow H^0(P, \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{L}^{a+b})$$

is surjective for $a, b \geq 1$.

Remark 3.7. The ampleness condition in Lemma 3.6 is that

$$H^1(P \times P, \mathcal{I}_\Delta \otimes \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \pi_1^* \mathcal{L}^a \otimes \pi_2^* \mathcal{L}^b) = 0 \quad (3.7)$$

for all $a, b \geq 1$, where \mathcal{I}_Δ is the ideal sheaf of the diagonal in $P \times P$.

Proposition 3.8. *Let P be a complete simplicial projective toric variety of dimension n and $\beta \in A_{n-1}(P)$ an ample Cartier degree. Then there is a $t_0 \in \mathbb{N}$ with the property that if E is the infinitesimal variation of Hodge structure associated to a generic $f \in S_{t\beta}$ for $t \geq t_0$, then $p_1(E) \neq \{0\}$.*

Proof. Let $\mathcal{E}_1 := \mathcal{O}_P$, $\mathcal{E}_2 := \Omega_P^n$, $a := 1$ and $1 \leq b \leq n-1$. If $[D] = \beta$ for an ample Cartier divisor D , then $\mathcal{L} := \mathcal{O}_P(tD)$ will satisfy condition (3.7) for all $t \geq t_0$ for some $t_0 \in \mathbb{N}$. Using Lemma 3.6 with $\mathcal{E}_1, \mathcal{E}_2, a$ and b as above and $\mathcal{L} := \mathcal{O}_P(tD)$ for $t \geq t_0$ we see that for a generic $f \in S_{t\beta}$

$$H^0(P, \mathcal{O}_P(X_f)) \otimes H^0(P, \Omega_P^n(bX_f)) \rightarrow H^0(P, \Omega_P^n((b+1)X_f))$$

is surjective. Then, by (3.2),

$$T_f B_\beta \otimes H^{n-p, p-1}(X_f)_v \rightarrow H^{n-p-1, p}(X_f)_v \quad (3.8)$$

is surjective. Therefore, the map obtained by iterating $n-1$ times (3.8)

$$(T_f B_\beta)^{\otimes (n-1)} \otimes H^{n-1, 0}(X_f)_v \rightarrow H^{0, n-1}(X_f)_v$$

is surjective. But, since by taking t_0 larger if necessary $h^{n-1, 0} \neq 0$ by Lemma 3.3, we obtain that $p_1(E) \neq 0$. \square

Theorem 3.9. *Let P be a complete simplicial projective toric variety of dimension $n \geq 4$ and $\beta \in A_{n-1}(P)$ the degree of an ample Cartier divisor. For $t \in \mathbb{N}$ let $E_{t\beta}$ be the infinitesimal variation of Hodge structure on $M_{t\beta}$ (see (3.1)) associated to a generic $f \in S_{t\beta}$. Then, there exists a $t_0 \in \mathbb{N}$ such that, if $E_{t\beta}$ satisfies*

$$\dim(p_0(E_{t\beta})) = \dim E_{t\beta}, \quad (3.9)$$

for $t \geq t_0$, $E_{t\beta}$ is non-generic in $V_{\dim E_{t\beta}}$.

Proof. This is immediate from Propositions 3.4 and 3.8, and Theorem 2.6. \square

Example 3.10. If $P := \mathbb{P}^n$, the infinitesimal Torelli and Macaulay's theorems imply that condition (3.9) holds. Fixing D as the hyperplane divisor, we conclude that the infinitesimal variation associated to hypersurfaces of sufficiently high degree are non generic, that is, we recover the result of [1].

We can now extend the result of the previous example to the case of variations of generic smooth hypersurfaces in smooth toric varieties. To show that in this case (3.9) holds we quote Theorem 0.1 from [10]:

Theorem 3.11. *Let P be a smooth complete algebraic variety of dimension $n \geq 2$ and X a smooth member of the linear system determined by a sufficiently ample line bundle on P . Then, the map*

$$H^1(X, \mathcal{T}_X) \rightarrow \text{hom}(H^0(X, \Omega_X^{n-1}), H^1(X, \Omega_X^{n-2}))$$

is injective.

Then, in the smooth case, we conclude from Theorems 3.9 and 3.11 that

Theorem 3.12. *Let P be a complete smooth projective toric variety of dimension $n \geq 4$ and $\beta \in A_{n-1}(P)$ the degree of an ample Cartier divisor. Let E be the infinitesimal variation of Hodge structure associated to a generic $f \in S_{t\beta}$. Then, there exists $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$, E is non-generic.*

Proof. Let t_0 be the value produced by Theorem 3.9, and X_f the hypersurface associated to a generic $f \in S_{t\beta}$ for $t \geq t_0$. Generically, X_f is smooth. Since by increasing t_0 if needed, $T_f M_{t\beta} \hookrightarrow H^1(X_f, \mathcal{T}_{X_f})$, by Theorem 3.11, we conclude that $E \simeq p_0(E)$ so that (3.9) holds and so does this result. \square

Remark 3.13. Assuming an infinitesimal Torelli theorem valid for (generic) hypersurfaces of sufficiently high degree in toric varieties, then it is possible to remove the smoothness requirement from Theorem 3.12. In the next section we show that smoothness is not an essential ingredient for the non-genericity result.

4. HYPERSURFACES IN WEIGHTED PROJECTIVE SPACES

Let $(q_0, \dots, q_n) \in \mathbb{N}^{n+1}$ such that $q_0 = \gcd(q_1, \dots, q_n) = 1$ and $m|s$ where $m := \text{lcm}(q_1, \dots, q_n)$ and $s := \sum_{j=0}^n q_j$.

Let $P := \mathbb{P}^n(q_0, \dots, q_n)$ be the weighted projective space with weights (q_0, \dots, q_n) (so that P is “well formed” by the conditions on the weights). P is a complete simplicial toric variety of dimension n with fan $\Sigma := \{\sigma_I : I \subset \{0, \dots, n\}\}$, where $\sigma_{\{i_1, \dots, i_k\}}$ is the cone generated by e_{i_1}, \dots, e_{i_k} , with $\{e_1, \dots, e_n\}$ the canonical basis of \mathbb{R}^n and $e_0 := -\sum_{j=1}^n e_j$. In particular, the 1-dimensional cones are $\Sigma^{(1)} := \{\sigma_j : j = 0, \dots, n\}$ where $\sigma_j := \mathbb{R}_{\geq 0} e_j$ and the n -dimensional cones are $\Sigma^{(n)} := \{\hat{\sigma}_j : j = 0, \dots, n\}$, where $\hat{\sigma}_j$ is the cone generated by all the e_l 's except

for e_j . The lattice $N \subset \mathbb{R}^n$ is generated by $\{f_1 := \frac{1}{q_1}e_1, \dots, f_n := \frac{1}{q_n}e_n\}$, and we define $f_0 := e_0 = -\sum_{j=1}^n q_j f_j$. Notice that f_j is the integral generator of σ_j for $j = 0, \dots, n$.

$A_{n-1}(P) \simeq \mathbb{Z}$ and if D_j is the torus-invariant divisor associated to the cone σ_j , $\deg(D_j) = q_j$. Hence the grading on $S := \mathbb{C}[z_0, \dots, z_n]$ is given by

$$\deg(z_0^{a_0} \cdots z_n^{a_n}) := \sum_{j=0}^n q_j a_j$$

Proposition 4.1. *Let $D := \sum_{j=0}^n a_j D_j$ be a Weil divisor in P with $d := \deg(D)$. Then, D is a Cartier divisor if and only if $m|d$. Also, D is ample if and only if $d > 0$.*

Proof. It is known (see [9], Exercise on page 62) that $\sum_{j=0}^n a_j D_j$ is Cartier if and only if for each maximal cone σ there is $u(\sigma) \in M$ such that for all f_j , lattice generators of the 1-dimensional cones contained in σ ,

$$\langle u(\sigma), f_j \rangle = -a_j. \quad (4.1)$$

If D is Cartier, there are $u(\widehat{\sigma}_k) \in M$ for each n -dimensional cone $\widehat{\sigma}_k$, $k = 0, \dots, n$ that satisfy (4.1). Evaluating each $u(\widehat{\sigma}_k)$ on the lattice generators f_j and using (4.1) it is easy to see that

$$u(\widehat{\sigma}_k) = - \left(\sum_{j=1}^n a_j f_j^* \right) + \alpha_k \quad \text{with} \quad \alpha_k := \begin{cases} 0 & \text{if } k = 0 \\ \frac{d}{q_k} f_k^* & \text{otherwise.} \end{cases} \quad (4.2)$$

Hence, since $u(\widehat{\sigma}_k) \in M$, $q_k|d$ for all k , and so $m|d$.

Conversely, if $m|d$, for each maximal cone $\widehat{\sigma}_k$ (4.2) defines an element of M that satisfies (4.1). Therefore, D is Cartier.

By Theorem 4.6 in [2], D is ample if and only if its piecewise linear support function ψ_D satisfies

$$\psi_D(f_{i_1} + \cdots + f_{i_k}) - (\psi_D(f_{i_1}) + \cdots + \psi_D(f_{i_k})) > 0 \quad (4.3)$$

for all primitive collections $\{f_{i_1}, \dots, f_{i_k}\}$. In our case, the only primitive collection is $\{f_0, \dots, f_n\}$, for which the left hand side of (4.3) evaluates to $\frac{d}{\max\{q_0, \dots, q_n\}}$ and the second assertion follows. \square

Corollary 4.2. *mD_0 is an ample Cartier divisor. In particular, P is a projective variety.*

Let E be the infinitesimal variation of Hodge structure associated to a generic hypersurface of degree $d = tm$ for some $t \in \mathbb{N}$. We are interested in the dimension of $p_0(E)$, which will be computed using the weighted Macaulay's theorem. Let us state it here, in a slightly simplified form. Let $\rho := (n+1)d - 2s$,

Theorem 4.3 (Weighted Macaulay's Theorem [19]). *If c is a multiple of m and $e \in \mathbb{N}$ such that $\rho - (c+e) > -s + mn$ then the natural map $R_e \rightarrow \text{hom}(R_c, R_{c+e})$ induced by the ring multiplication is injective.*

We can then derive the following

Corollary 4.4. *If $d \geq 2m$ we have $\dim p_0(E) = \dim E$.*

Proof. To determine $\dim p_0(E)$ we apply Theorem 4.3 for $e = d$ and $c = d - s$. Then our assumptions ensure that m divides c . Moreover

$$\rho - (c + e) = (n + 1)d - 2s - d - d + s = (n - 1)d - s$$

then as $d \geq 2m$ we have that $\rho - (c + e) \geq 2m(n - 1) - s \geq mn - s$. The weighted Macaulay theorem then gives us that $R_d \hookrightarrow \text{hom}(R_{d-s}, R_{2d-s})$. Using the diagram (3.3) this gives us the following injection:

$$E \hookrightarrow \text{hom}(H^{n-1,0}, H^{n-2,2})$$

which exactly means that $p_0(E) \simeq E$. \square

Theorem 4.5. *Let $P = \mathbb{P}^n(q_0, \dots, q_n)$ with $n \geq 4$, $q_0 = \gcd(q_1, \dots, q_n) = 1$. Then, if $m|d$ and $m|s$, there is a degree d_0 such that for all $d \geq d_0$ the infinitesimal variation of Hodge structure associated to a generic hypersurface of degree d is non-generic.*

Proof. Recall that E is the infinitesimal variation of Hodge structure associated to a generic hypersurface of degree tm . By Corollary 4.4, if $t \geq 2$ condition (3.9) holds. On the other hand, since m is the degree of an ample Cartier divisor, by Theorem 3.9 the result follows. \square

5. COMPLETE INTERSECTIONS

In this section we prove that the infinitesimal variations of Hodge structure associated to generic complete intersections of sufficiently high degree in \mathbb{P}^n are non-generic. The context is the following: we have $X \subset \mathbb{P}^n$ given as the complete intersection of c hypersurfaces of degrees $d_1 \geq \dots \geq d_c$, each determined by a homogeneous polynomial F_a for $a = 1, \dots, c$. We will assume that X is smooth. The degree of the canonical bundle of X , K_X , will be denoted by $d(X)$, so that, $d(X) := \sum_{a=1}^c d_a - (n + 1)$. In this section we use the *primitive* Hodge numbers, that is, $h^{p,q} := \dim H^{p,q}(X)_p$.

The following Theorem states the relevant known results on the cohomology of a complete intersection.

Theorem 5.1. *With the notation as above, let $S := \mathbb{C}[x_0, \dots, x_n, \mu_1, \dots, \mu_c]$, $F := \sum_{a=1}^c \mu_a F_a \in S$, $J := \langle \frac{\partial F}{\partial \mu_1}, \dots, \frac{\partial F}{\partial \mu_c}, \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n} \rangle$ the Jacobian ideal of F and $R := S/J$ seen as a bigraded ring with the grading induced by $\deg(x_j) := (0, 1)$ and $\deg(\mu_a) := (1, -d_a)$. Then:*

- (1) $H^{n-c-p,p}(X)_p \simeq R_{(p,d(X))}$, where H_p is the primitive cohomology. In particular, $H^{n-c,0}(X)_p \simeq R_{(0,d(X))}$ and $H^{n-c-1,1}(X)_p \simeq R_{(1,d(X))}$.
- (2) $H^1(X, \mathcal{T}_X) \simeq R_{(1,0)}$, where \mathcal{T}_X is the tangent sheaf of X . Also, if $d(X) \geq 0$, the tangent space to the moduli of X is isomorphic to $H^1(X, \mathcal{T}_X)$.
- (3) The diagram

$$\begin{array}{ccc} H^1(X, \mathcal{T}_X) & \longrightarrow & \text{hom}(H^{n-c-p,p}(X)_p, H^{n-c-p-1,p+1}(X)_p) \\ \simeq \downarrow & & \downarrow \simeq \\ R_{(1,0)} & \longrightarrow & \text{hom}(R_{(p,d(X))}, R_{(p+1,d(X))}) \end{array} \quad (5.1)$$

—where the top horizontal arrow is the action of the infinitesimal variation and the bottom one is induced by multiplication— commutes up to a multiplicative constant.

Part (1) is due to Terasoma, Konno, Dimca [7, §2]. The first part of (2) and part (3) are obtained by adapting Terasoma's proofs of Propositions 2.5 and 2.6 in [18] to the bigraded context. These results are proved using a *Cayley trick*, that is, by associating to X a hypersurface $\mathcal{X} := (F = 0)$ in a larger ambient space and studying the relation between the cohomologies of X and \mathcal{X} . The second assertion of (2) is Theorem 3.5 in [16].

Remark 5.2. The complete intersection assumption ensures that the sequence (F_1, \dots, F_c) is regular.

In order to apply Theorem 2.6, we need to meet condition (2.4) and check that the conditions on the projections of the infinitesimal variation hold. The purpose of the following result is to solve the first issue.

Proposition 5.3. *Assume that $\dim X = n - c \geq 2$. Then there is $K_1 \in \mathbb{N}$ such that, for $d(X) \geq K_1$,*

$$\mu \geq 3 \left(\left\lceil \frac{h^{n-c-1,1} - 1}{h^{n-c,0}} \right\rceil + 1 \right), \quad (5.2)$$

where μ is the dimension of the infinitesimal variation of Hodge structure associated to the family of deformations of X .

Proof. This proposition follows readily from polynomial estimates of the different dimensions that appear in (5.2).

Using combinatorics, it is not difficult to show the following bounds:

$$\begin{aligned} h^{n-c-1,1} &\leq \frac{c}{n!} d(X)^n + O(d(X)^{n-1}) \\ \dim H^1(X, \mathcal{T}_X) &\geq \frac{1}{n!c^n} d(X)^n + O(1) \end{aligned}$$

and, for $c \geq 2$,

$$h^{n-c,0} \geq \frac{1}{n^n} d(X)^{n-c}$$

while, for $c = 1$,

$$h^{n-1,0} \geq \frac{1}{n!} d(X)^n.$$

For $c \geq 2$, the right hand side of (5.2) is bounded from above by $3 \frac{n^n c 2^n}{n!} d(X)^c + O(d(X)^{c-1})$, and, $\mu = \dim H^1(X, \mathcal{T}_X) \geq \frac{1}{n!c^n} d(X)^n + O(1)$. Then, since $c < n$, the statement follows. The case for $c = 1$ is the analogous. \square

Remark 5.4. It is possible to give an effective version of Proposition 5.3. Indeed, if

$$d(x) \geq \begin{cases} \max\{n, \sqrt[n-c]{3n^n c^{n+1} 2^n + 1}, \sqrt[n-c]{n! c^n (3 + c^2 + (n+1)^2)}\}, & \text{if } c \geq 2, \\ \max\{n, \sqrt[n]{n! (2^n 3 + 4 + (n+1)^2)}\}, & \text{if } c = 1 \end{cases}$$

then (5.2) and, Theorem 5.6 hold. Since we don't find this explicit bound very illuminating, or useful, we omit the details.

Next we prove that the conditions on the projections of the infinitesimal variation required by Theorem 2.6 hold.

Proposition 5.5. *There is a $K_2 \in \mathbb{N}$ such that if E is a complete intersection infinitesimal variation of Hodge structure with $d(X) \geq K_2$, then we have:*

$$p_0(E) \simeq E \quad \text{and} \quad p_1(E) \neq \{0\}$$

Proof. First we refer to the local Torelli theorem proved by C. Peters in [16]. There he proved that the following map is injective:

$$\begin{aligned} E &\hookrightarrow \operatorname{hom}(H^{n-c,0}(X)_p, H^{n-c-1,1}(X)_p) \\ \alpha &\longmapsto \alpha|_{H^{n-c,0}(X)_p} \end{aligned}$$

This means exactly that $E \simeq p_0(E)$. Next, using Lemma 3.4 of J. Nagel [14], we see that the map

$$R_{(1,0)} \otimes R_{(1,d(X))} \longrightarrow R_{(2,d(X))}$$

induced by multiplication is surjective, so using diagram (5.1) we find that the corresponding map in cohomology $E \otimes H^{n-c-1,1}(X)_p \rightarrow H^{n-c-2,2}(X)_p$ is surjective; therefore $p_1(E) \neq \{0\}$ as soon as $h^{n-c-2,2}$ is non-zero. But using the surjectivity of the multiplication in R and diagram (5.1) once more, the following map is also surjective:

$$E \otimes H^{2,n-c-2}(X)_p \longrightarrow H^{1,n-c-1}(X)_p.$$

Now, as seen in the proof of Proposition 5.3, $\dim H^1(X, \mathcal{T}_X) \geq \frac{1}{n!c^n} d(X)^n + O(1)$, so that there is $K_2 \in \mathbb{N}$ with the property that $H^1(X, \mathcal{T}_X) \neq 0$ if $d(X) \geq K_1$. If this last condition holds, then $E \neq \{0\}$. Therefore, $h^{1,n-c-1} = h^{n-c-1,1}$ is non-zero because E injects into $\operatorname{hom}(H^{n-c,0}(X)_p, H^{n-c-1,1}(X)_p)$; this then implies that $h^{n-c-2,2} = h^{2,n-c-2} \neq 0$. \square

From Theorem 2.6, whose hypotheses are satisfied because of Propositions 5.5 and 5.3, we derive the final result:

Theorem 5.6. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection defined by the intersection of c hypersurfaces of degrees d_1, \dots, d_c . If $\dim X \geq 3$ there exists $K \in \mathbb{N}$ such that if the degree of the canonical bundle $d(X) = \sum_{a=1}^c d_a - (n+1) \geq K$, then the infinitesimal variation of Hodge structure associated to the deformation of X , E , is non-generic in $V_{\dim E}$.*

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